THE MARCH MEETING OF THE AMERICAN MATHEMATICAL SOCIETY AT CHICAGO.

THE twelfth regular meeting of the American Mathematical Society at Chicago, being also the forty-third regular meeting of the Chicago Section, was held on Friday and Saturday, March 28 and 29, at the University of Chicago. The various sessions were attended by about forty persons, among whom were the following thirty-one members of the Society:

Dr. I. A. Barnett, Professor H. F. Blichfeldt, Professor G. A. Bliss, Dr. Henry Blumberg, Professor W. H. Bussey, Professor R. D. Carmichael, Professor E. W. Chittenden, Professor A. R. Crathorne, Professor D. R. Curtiss, Professor L. E. Dickson, Professor L. W. Dowling, Mr. E. B. Escott, Professor A. M. Kenyon, Professor W. C. Krathwohl, Professor Kurt Laves, Professor A. C. Lunn, Professor G. A. Miller, Professor E. H. Moore, Professor E. J. Moulton, Professor H. L. Rietz, Professor W. H. Roever, Mr. J. B. Rosenbach, Dr. A. R. Schweitzer, Professor J. B. Shaw, Professor E. B. Skinner, Professor H. E. Slaught, Professor E. B. Van Vleck, Professor G. E. Wahlin, Professor E. J. Wilczynski, Professor J. W. A. Young, and Professor Alexander Ziwet.

Twenty-nine persons joined in a dinner at the Quadrangle Club on Friday evening. Following the dinner, a number of short informal speeches were made in response to calls from Professor Bliss who presided. Professor Slaught read interesting extracts from a letter which he had just received from Professor E. R. Hedrick, who is in France helping to organize educational work for members of the American Expeditionary Forces. Professor Ziwet of the University of Michigan extended a most cordial invitation to members of the Section to come to the summer session of the Society next September at Ann Arbor. Several members spoke against the too common practice among mathematicians of giving gratuitously information of a technical nature to persons untrained in mathematics; it was argued that the worth of mathematics would be more widely appreciated and its development correspondingly more rapid if mathematicians demanded adequate compensation for such services.

At a short business session on Saturday morning there was

a discussion of the desirability of holding the December meeting of the Section this year in St. Louis in conjunction with the meetings of the American association for the advancement of science. It was voted to leave the matter to the program committee for decision.

In reading his paper on "Differential corrections for anti-aircraft guns," Professor Bliss paid tribute to the efforts of mathematicians who have been engaged in war work of a mathematical nature, mentioning especially the effectiveness of the work headed by Major F. R. Moulton at Washington and by Major Oswald Veblen at the Aberdeen Proving Ground on ballistic problems.

Friday afternoon was devoted to a symposium on the geometry of numbers, relating largely to the work of Minkowski. Formal papers were presented as follows:

I. Professor H. F. BLICHFELDT: "Report on the theory of the geometry of numbers," giving the fundamental theorems with applications to homogeneous and non-homogeneous forms.

II. Professor L. E. DICKSON: "Applications of the geometry of numbers to algebraic numbers."

Synopses of these papers will appear in the July BULLETIN.

Professor G. A. Bliss, chairman of the Section, presided at the various sessions, being relieved by Professors Curtiss, Miller, and Carmichael. At the sessions on Friday and Saturday mornings the following papers were read:

(1) Professor E. J. WILCZYNSKI: "A new geometrical representation for a function of a complex variable."

(2) Dr. A. L. NELSON: "Plane nets conjugate to a given congruence of straight lines" (preliminary report).

(3) Professor A. B. COBLE: "On the ten nodes of a rational plane sextic and of the Cayley symmetroid."

(4) Dr. FLORENCE E. ALLEN: "On a class of sectrix curves."

(5) Professor W. H. BUSSEY: "A note on the problem of the eight queens."

(6) Professors E. W. CHITTENDEN and A. D. PITCHER: "On the theory of developments of an abstract class in relation to the calcul fonctionnel."

(7) Professor E. J. WILCZYNSKI: "The scientific work of Gabriel Marcus Green."

(8) Professor G. A. BLISS: "Differential corrections for anti-aircraft guns."

(9) Professor H. L. RIETZ: "Functional relations for which the coefficient of correlation is zero."

(10) Professor L. L. DINES: "Systems of linear inequalities."

(11) Professor E. J. MOULTON: "A note on a proof of the law of the mean."

(12) Dr. A. R. SCHWEITZER: "On pseudo-groups with an application to the algebra of logic."

(13) Professor G. A. MILLER: "Groups containing a relatively large number of operators of order two."

(14) Dr. HENRY BLUMBERG: "On a theorem of W. H. Young and G. C. Young."

(15) Professor E. B. VAN VLECK: "On Green's lemma."

In the absence of the authors, the papers of Dr. Nelson, Professor Coble, Dr. Allen and Professor Dines were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. In this paper, Professor Wilczynski interprets the complex variables z and w as points upon the same Neumann sphere, and joins by straight lines the points which correspond to each other by means of a functional relation $w = F(z)$. The resulting congruence has a large number of interesting properties. Its focal surfaces are real surfaces of positive curvature, and the foci on each line of the congruence divide harmonically its two points of intersection with the sphere. The congruence is a W-congruence, and all of the asymptotic lines of the focal surface belong to linear complexes. The Neuman sphere is the directrix quadric of the focal surface. The asymptotic lines of the focal surface correspond to the minimal lines of the z-plane and can be obtained with ease. The developables of the congruence, which are real, are also obtained and interesting relations are shown to exist between the directrices, axes, and rays of the two focal sheets, as well as between the directrix, axis, and ray curves.

2. Dr. Nelson has made use of Wilczynski's methods in the study of the projective properties of a congruence of straight lines as related to those of the plane nets which are conjugate to it, in the sense of Guichard. Following are some of the results.

(1) If the plane nets have straight lines for constant values of one parameter, one of the nappes of the focal surface of the congruence is developable, and conversely.

(2) If the first Laplace transforms of the plane nets have straight lines corresponding to the constant values of one parameter, the new focal nappe of the minus first derived congruence becomes developable, and conversely.

(3) If the plane nets are periodic under the Laplace transformation, of period three, the following properties hold for the congruence:

(*a*) The lines which join the first and second, and the minus first and minus second, Laplace transforms of the surface point of either of the focal nappes will intersect on the axis of that surface point.

(*b*) This point of intersection will be the harmonic conjugate of the surface point with respect to the foci of its axis. It will also be one of the foci of the joint axis which corresponds to the surface point.

(*c*) The joint axis curves and joint ray curves coincide and become parametric.

(*d*) The parametric tangents at a particular surface point of either focal nappe will be the double rays of the involution determined by the axis tangents and ray tangents at the same point.

3. The notion as developed by Professor Coble of the congruence of n points in a projective space under Cremona transformation leads to an infinite discontinuous group which can be exhibited as a group of linear transformations with integer coefficients. Thus the group has a modular theory and in an earlier paper the types of finite groups obtained by reduction modulo 2 were determined. It now appears that these modular groups have not merely an arithmetic but also a geometric existence when the n points are properly specialized. Such cases for example are the ten nodes of a rational plane sextic, and in space the ten nodes of a Cayley symmetroid. Some of the theorems obtained are as follows. A general rational plane sextic can be transformed into precisely $2^{13} \cdot 31 \cdot 51$ projectively distinct sextics and under such transformation these types are permuted by a group isomorphic with a theta modular group of genus 5. Again if there exists a quartic curve with a triple point at one node

and on the nine others (one condition on the sextic) then there exists a similar quartic with a triple point at *any* one of the nodes. Similar theorems apply to the symmetroid. In this case the genus of the group is 4. There is thus foreshadowed a uniform parametric representation of the ten nodes in terms of abelian modular functions of genus four.

4. A system of curves discussed in a paper presented to the Society by Dr. James H. Weaver in April, 1917, is shown by Miss Allen to be a special case of the Schoute sectrix curves and of the araneïd or "spider" curves. It is shown that the system for successive values of n may be generated by a birational transformation, consisting of an inversion followed by a perspectivity. The generation of the entire system of araneïds by alternate applications of these two operations is also proved.

5. The problem of the eight queens is the determination of the ways in which eight queens can be placed on a chess board—or more generally, in which n queens can be placed on a square board of n^2 cells—so that no queen can take any other queen. Professor Bussey's note shows that in the special case in which n is a prime number p, there are $p^2 - 3p$ solutions furnished by the $p^2 - 3p$ straight lines of slopes 2, 3, 4, $\cdots$, $(p - 2)$ of the finite euclidean geometry of order p^2. The solutions furnished by this method are not the only solutions, however.

6. In a paper, which has not been published, entitled "Classes which admit a development" (presented to the Society, March 21, 1913), Professors Chittenden and Pitcher discussed the theory of developments Δ of an abstract class in relation to the calcul fonctionnel of Fréchet. [A development Δ of a class P is a system $((P^{m_i}))$ of subclasses of P $(m = 1, 2, 3 \cdots; l^m = l_1^m, l_2^m, \cdots)$. Cf. E. H. Moore, Introduction to a Form of General Analysis, Yale University Press, New Haven, 1910.] In the present paper the above mentioned theory is refounded and based upon a set of five properties of a development Δ which together imply that the class P is a compact metric space (cf. Hausdorff, Grundzüge der Mengenlehre, Leipzig, 1914). The five properties are proved to be completely independent in the sense of E. H. Moore (loc. cit.).

The results of the general theory are applied to obtain the necessary and sufficient conditions that a topological space (cf. Hausdorff, loc. cit.) be equivalent to a compact metric space. A further application is made to spaces of non-metrical analysis situs. Every limited domain in a space δ satisfying the axiom system Σ_1 (Σ_2) of R. L. Moore (*Transactions of the American Mathematical Society*, volume 17, 1916) is a compact metric space. This result was obtained independently by R. L. Moore by another method.

The paper will appear in the *Transactions*.

7. Professor Wilczynski's paper, intended for publication in the BULLETIN, is devoted to a brief description of the mathematical work of Gabriel Marcus Green, whose premature death inflicts a heavy loss on American mathematics.

8. In the applications of ballistics it is customary to compute a first approximation to the path of a projectile on the assumption that there is no wind, and on the further assumptions that the density of the air and the weights of the projectile and powder charge are normal. These are ideal conditions rarely if ever realized in actual firing. The range table must therefore contain differential corrections which can be applied in the field, either mechanically or personally, to account for the actual conditions at the time when the firings are made. The so-called differential corrections appear as the solutions of a system of linear differential equations which are solved by mechanical quadratures. In the paper of Professor Bliss an auxiliary system of linear differential equations is used by means of which the number of solutions to be computed by mechanical quadratures is reduced to three. The remainder of the computation is of the simplest sort, requiring no especial skill, and relatively rapid.

9. In this paper, Professor Rietz treats some simple and striking cases in which the correlation coefficient of two variables x and y is zero although there is a simple functional relation between x and y. The result $r = 0$ may be accounted for by a consideration of the lines of regression. This brings out the importance of testing the curve of regression, and the danger of making applications of the correlation coefficient without knowledge of the underlying theory. In the cases

cited, for a single valued function $y = f(x)$, the correlation ratio η_y can be used appropriately instead of r to give a summary notion of correlation.

10. In this paper, Professor Dines constructs a theory relative to systems of linear inequalities, which to a considerable extent parallels the classic theory of systems of linear equations. The paper will appear in the *Annals of Mathematics*.

11. A proof of the law of the mean for derivatives has been based on the law of the mean for integrals (Goursat-Hedrick, Mathematical Analysis, volume I, page 155). Professor Moulton calls attention to an error in the proof, and shows how it can be corrected. The hypotheses of the theorem proved in this way are more restrictive than when proved in another common way. It is pointed out that if Denjoy's generalization of integration is used the derivative $f'(x)$ need not be continuous on the interval (a, b), nor integrable in the sense of Riemann or Lebesgue. The restrictions are that $f'(x)$ must exist and be finite at every point of the interval.

12. A set E of elements forms a pseudo-group if E satisfies a set of properties $\Sigma[\rho_1, \rho_2, \cdots]$ necessary for an abstract group (finite or infinite) and including closure with regard to the undefined relations $\rho_1, \rho_2, \cdots$ generating Σ. Every group is a pseudo-group as are also the semi-groups of Dickson and Hilton. In a previous paper Dr. Schweitzer exhibited a pseudo-group $\Sigma(\theta)$ satisfied by $\theta(x, y) = x^{-1}y^{-1}$ of an abstract group. In the present paper Dr. Schweitzer constructs further pseudo-groups, in particular with reference to distributive properties. Among the latter class there exists a pseudo-group satisfied by $\zeta(x, y) = x^{-1}yx$ and $\tau(x, y) = xyx^{-1}$ of an abstract group and possessing duality with reference to the generating relations $\zeta(x, y)$ and $\tau(x, y)$. This pseudo-group $\Sigma[\zeta, \tau]$ has the property that it is satisfied by any set of elements obeying the "additive" and "multiplicative" laws of the algebra of logic. Incidentally, a new set of postulates for the latter discipline is obtained merely by assuming that for the pseudo-group $\Sigma[\zeta, \tau]$ the "principle of absorption" $\zeta[x, \tau(x, y)] = x$ is valid. Fundamental for the discussion is

the fact that the properties $\zeta(x, x) = x$, $\zeta[x, \tau(x, y)] = x$ or y, $\zeta[x, \tau(y, z)] = \tau[\zeta(x, y), \zeta(x, z)]$ and their duals are valid both for the abstract group $[\zeta(x, y) = x^{-1}yx, \tau(xy) = xyx^{-1}]$ and the algebra of logic $[\zeta(x, y) = x \cdot y, \tau(x, y) = x + y]$.

13. Professor Miller's paper appears in full in the present number of the BULLETIN.

14. A given planar set S is said to have a triangular void at a point P, if P is the vertex of a triangle containing no points of S in its interior. According to a recent result of W. H. and G. C. Young (*Proceedings of the London Mathematical Society*, 1918), a planar set having a triangular void at every one of its points is of measure zero. Dr. Blumberg gives a simple proof of this theorem, and a generalization based on the following definition: Let C_r be the circle of center P and radius r and $m_e(S, C_r)$ the exterior measure of the portion of S in C_r; then we define the "upper exterior measure of S at P" as

$$\limsup_{r \to 0} \frac{m_2(S, C_r)}{\pi r^2}.$$

The generalization is as follows: A planar set having its upper exterior measure < 1 at every one of its points is of measure zero. This theorem is utilized to obtain certain general properties of functions of two variables. Extension to higher spaces is immediate.

15. In the proofs which have been given for Green's fundamental lemma

$$\int\int \frac{\partial P(x, y)}{\partial x} dxdy = \int_c P(x, y)dy$$

notable restrictions have been imposed upon the boundary. The usual condition is that the contour C shall be cut by a parallel to the axis of X in a finite number of points or consist of a finite number of "regular" pieces. The paper of Professor Van Vleck gave a proof for the general case in which C is merely conditioned to be a simple closed rectifiable curve. Over its closed interior $P(x, y)$ is supposed to be continuous, while the requirement for $\partial P/\partial x$ is that it shall be properly integrable.

E. J. MOULTON,
Acting Secretary of the Chicago Section.

THE APRIL MEETING OF THE SAN FRANCISCO SECTION.

THE thirty-third regular meeting of the San Francisco Section was held at Stanford University on Saturday, April 5, with the chairman, Professor Cajori, presiding. The total attendance was twenty, including the following thirteen members of the Society:

Professor R. E. Allardice, Professor B. A. Bernstein, Professor Thomas Buck, Professor Florian Cajori, Professor M. W. Haskell, Professsor L. M. Hoskins, Professor D. N. Lehmer, Professor W. A. Manning, Professor H. C. Moreno, Dr. F. R. Morris, Professor C. A. Noble, Dr. Pauline Sperry, Dr. J. S. Taylor.

The next meeting of the Section will be held at the University of California, October 25, 1919.

The following papers were presented at the April meeting:

(1) Professor R. E. ALLARDICE: "Note on the sextic with eight cusps."

(2) Professor FLORIAN CAJORI: "Observations on the development of algebraic notations."

(3) Dr. J. S. TAYLOR: "A set of five postulates for Boolean algebras in terms of the operation 'exception.'"

(4) Dr. PAULINE SPERRY: "On the work of Gabriel Marcus Green in the field of projective differential geometry."

(5) Dr. Y. R. CHAO: "A note on 'continuous' mathematical induction."

(6) Professor L. M. HOSKINS: "Effect of the asssumption of variable elasticity upon estimates of the rigidity of the earth."

(7) Professor W. A. MANNING: "A fundamental theorem for simply transitive primitive groups."

(8) Professor E. T. BELL: "On the number of representations of an integer as a sum of 3, 5, 7, 9, 11, or 13 squares."

(9) Professor E. T. BELL: "On the number of representations of $2n$ as a sum of $2r$ squares."

(10) Dr. L. L. SMAIL: "Summability of double series."

Dr. Chao was introduced by Professor Bernstein. The papers of Professor Bell and Dr. Smail were read by title. Abstracts of the papers follow.

1. In his note on the sextic with eight cusps, Professor Allardice finds the equation of this curve, and establishes the relations that connect the eight cuspidal points.

2. Professor Cajori observes that the adoption of algebraic notations is subject to two opposing forces: (1) The disinclination of mathematicians to memorize many symbols, (2) The convenience and power acquired from a well-chosen symbolism. In elementary algebra no large group of symbols invented by any one author has ever been bodily and permanently adopted; each symbol had its own struggle for existence, dependent not only upon merit but also upon particular configuration of circumstances. Bi-asymmetrical marks other than the numerals and letters of alphabets hardly ever survived. Notations have crossed national boundaries less easily than have principles and processes. There still exist side by side certain duplicate notations, rendering elementary algebra unnecessarily difficult. There is little precedent to indicate how much in the acquirement of desirable notations can be achieved by national and international organization.

3. Dr. Taylor presents a set of five comparatively simple postulates in terms of the operation "exception" with a proof of their consistency, independence, and sufficiency for the logic of classes. The complete existential theory of the five postulates is then developed, followed by a short discussion of the relative importance of the element 1 and negation (not-a). In order to show that each is as powerful as the other in one respect at least, a second set of postulates is presented in which, instead of postulating the element 1 and defining not-a, not-a is postulated and the element 1 defined, a thing which curiously enough has never been done heretofore.

4. In a brief résumé, Dr. Sperry recited some of Dr. Green's more important contributions to the theory of projective differential geometry, emphasizing especially his work on (a) the geometry of one-parameter families of space curves, and conjugate nets on a curved surface, and (b) the general theory of curved surfaces and rectilinear congruences, as best illustrating the uniqueness of his analytic methods and the keenness of his geometric intuitions.

5. Dr. Chao proves a theorem concerning a propositional function $\varphi(x)$ defined for a real interval and satisfying the following two conditions: (1) $\varphi(a)$ is true; (2) there exists a constant Δ in the interval such that $\varphi(x)$ implies $\varphi(x \pm \delta)$ for $0 < \delta \leqq \Delta$.

6. Estimates of the rigidity of the earth are based upon the comparison of the actual yielding of the earth to tidal and centrifugal forces, as inferred from certain refined measurements, with the computed yielding of an elastic sphere having the dimensions and mass of the earth. The solution of this problem is known for a compressible sphere of uniform density, and for an incompressible sphere in which the density varies with distance from the center; the elastic moduli being in both solutions assumed to have uniform values throughout the body. Professor Hoskins has obtained the solution for a class of cases in which the elastic moduli as well as the density vary with distance from the center, and has obtained numerical results for a series of cases conforming closely with what is known regarding the actual properties of the earth. It is found that the known observational results are in close harmony with the computed strain of a sphere in which the elastic moduli are assumed to vary continuously from surface values based upon the known properties of rocks to values between six and seven times as great at the center. This would give a value of the modulus of rigidity at the center of about 1.70×10^{12} C. G. S. units—about double the value for steel.

7. The theorem stated by Professor Manning is to this effect: If the subgroup that leaves one letter of a simply transitive primitive group fixed has a multiply transitive constituent, of degree m say, then one of two things must occur; either all the transitive constituents of the subgroup that fixes one letter are multiply transitive groups of the same degree m in simple isomorphism, or at least one transitive constituent of the maximal subgroup in question is of a degree greater than m and a divisor of $m(m-1)$.

8. Defining as a primitive function of a positive integer n one whose values may be calculated from the real divisors of n alone, and as a simple function any sum of a finite number of primitive functions whose arguments form a recurring

series of the second order, Professor Bell determines all cases in which the number of representations of an integer as a sum of 3, 5, 7, 9, 11, or 13 squares is simple. The integers to be represented are classified first according to their linear forms modulo 8. For 3, 5, 7, 9 squares all possible cases are simple; for 11 or 13 squares only some are simple, and for 15, 17, 19, 21, 23, 25 squares none are simple. The criteria deciding this seem to indicate non-simplicity for all odd numbers of squares > 25; hence a special interest attaches to the cases treated in the paper. The resulting formulas are well adapted to numerical computation, and a comparison with the existing formulas for 5 or 7 squares (Eisenstein, H. J. S. Smith) shows a marked saving of labor in actual use. For 3, 5 and 7 squares, and for some cases of 9, 11 or 13, the number of representations may be calculated by several rapid recurrences. An interesting by-product of the determination for 3 squares is a new derivation of the class number for a negative determinant by finite processes only. This leads to results, in finite form, which also easily yield numerical values, and which should be of use in the construction of tables beyond those existing, should the necessity for such arise.

9. In two papers Liouville (*Journal de Mathématiques*, series 2, volume 6, 1861, pages 233,369) gave four general formulas relating to the number of representations of $2n$ as a sum of $2r$ squares. These contain as very special cases all the known instances in which such representations are either primitive or simple as defined in the preceding paper. Liouville suppressed both the proofs and the essential detail of showing how the undetermined coefficients, without knowledge of which his formulas cannot be used, may be found; these are supplied in Professor Bell's second paper. In proving the formulas it appears that similar general theorems exist for $2n + 1$ as a sum of $2r$ squares, and that corresponding but less elegant results may be stated for both $2n$ and $2n + 1$ as a sum of $2r + 1$ squares. Total and proper (as defined by Eisenstein) representations are treated by one analysis.

10. In a recent paper in the *Annals of Mathematics* (December, 1918), Dr. Smail has given a general method of summation for divergent series, based on a summation formula involving a function undefined except for certain restrictive

conditions. In the present paper he extends this method to double series, and discusses the application of the method to convergent double series. It is found that the summation functions of the familiar methods of Cesàro, Hölder, Borel, LeRoy, Riesz, Vallée-Poussin, etc., can be used in building up summation formulas for double series.

B. A. BERNSTEIN,
Secretary of the Section.

ON A CERTAIN GENERATION OF RATIONAL CIRCULAR AND ISOTROPIC CURVES.

BY PROFESSOR ARNOLD EMCH.

(Read before the American Mathematical Society December 28, 1917.)

1. *Introduction.*

A CIRCULAR curve contains the circular points at infinity, or the isotropic points of the plane, as single, or as singular points. A plane isotropic curve is defined as a curve, all of whose infinite points are absorbed by the isotropic points. The equation of such a curve, which is necessarily of even order, in cartesian coordinates may be written in the form

$$(x^2 + y^2)^k + \varphi(x, y) = 0, \tag{1}$$

in which $\varphi(x, y)$ is a polynomial of degree $2k - 1$ at most.

If $P(\xi, \eta)$ is a fixed point and $A(x, y)$ any other point so that $PA = \rho$, and θ the angle between PA and the positive direction of the x-axis, then the coordinates of A are $x = \xi + \rho\cos\theta$, $y = \eta + \rho\sin\theta$, and satisfy equation (1) when A is on the curve. The condition for this is an equation of the form

$$\rho^{2k} + \alpha_1\rho^{2k-1} + \alpha_2\rho^{2k-2} + \cdots + \alpha_{2k-1}\rho + \alpha_{2k} = 0, \tag{2}$$

in which $\alpha_1, \alpha_2, \cdots, \alpha_{2k-1}$ are coefficients, which, in general, depend on ξ, η, θ and the coefficients of (1); while α_{2k} is independent of θ. The roots $\rho_1, \rho_2, \cdots, \rho_{2k}$ of (2) are the distances PA_i $(i = 1, 2, 3, \cdots, 2k)$ of the points of inter-

section $A_1, A_2, \cdots, A_{2k}$ of the transversal through P, including an angle θ with the positive part of the x-axis. Evidently the product

$$\rho_1\rho_2 \cdots \rho_{2k} = \alpha_{2k} \tag{3}$$

is constant for all transversals through P.

This constant product is called the power of the point P with respect to the isotropic curve. This is one of the principal properties* of an isotropic curve. In fact an isotropic curve may also be defined by this property. In this paper I shall establish the necessary and sufficient conditions for the form of parametric representation of rational circular, in particular of rational isotropic curves, and their generation by rational transformations in a complex plane. As an application the complete representation of all rational circular cubics, and all rational isotropic quartics will be given.

2. *Parametric Representation of Rational Circular and Isotropic Curves.*

Let

$$x = \frac{F(t)}{H(t)}, \qquad y = \frac{G(t)}{H(t)}, \tag{4}$$

in which F, G, H are polynomials in t, with real coefficients, without a common factor, represent a rational curve. To real values of t correspond real values of x and y, or real points on the curve. The curve passes once through each of the isotropic points, when for a complex value t_i of t

$$\lim_{t=t_i}(x) = \infty, \qquad \lim_{t=t_i}(y) = \infty, \qquad \lim_{t=t_i}\left(\frac{y}{x}\right) = +i, \tag{5}$$

and, as a consequence, also for the conjugate value $\bar{t}_i$

$$\lim_{t=\bar{t}_i}(x) = \infty, \qquad \lim_{t=\bar{t}_i}(y) = \infty, \qquad \lim_{t=\bar{t}_i}\left(\frac{y}{x}\right) = -i. \tag{6}$$

The algebraic sign of i might, of course, be reversed. From this it follows that a necessary condition that the curve (4) pass through the isotropic points is that $H(t) = 0$ has complex roots which are not common to $F(t) = 0$ and $G(t) = 0$. But

* E. Pascal, Repertorium der höhern Mathematik, vol. 2 (first half), pp. 436–438 (1910).

complex roots of $H(t) = 0$ occur in conjugate pairs. Let $t_1, \bar{t}_1;\ t_2, \bar{t}_2;\ \cdots;\ t_k, \bar{t}_k$ be complex roots of this equation, so that $H(t)$ may be written in the form

$$(7)\quad H(t) = [(t-t_1)(t-t_2)\cdots(t-t_k)][(t-\bar{t}_1)(t-\bar{t}_2)\cdots(t-\bar{t}_k)]\Phi(t),$$

where $\Phi(t)$ is a polynomial in t, which contains none of the other factors of $H(t)$. The two brackets in (7) may be written in the form

$$t^k + \alpha_1 t^{k-1} + \alpha_2 t^{k-2} + \cdots + \alpha_k,$$

and

$$t^k + \bar{\alpha}_1 t^{k-1} + \bar{\alpha}_2 t^{k-2} + \cdots + \bar{\alpha}_k,$$

and are clearly conjugate expressions, so that the first may be written in the form $r - is$, the second in the form $r + is$, in which r and s are in general polynomials in t of degree k. Hence, (7) has the form

$$(8)\quad H(t) = (r^2 + s^2)\Phi(t).$$

Incidentally we have proved

THEOREM 1: *If a polynomial equation with one unknown, and real coefficients, has only imaginary roots, then the polynomial may be represented as the sum of the squares of two other polynomials.*

Suppose now that t_i satisfies the equation $r - is = 0$, and that for this value of t not both r and s vanish simultaneously. Then t_i, according to (5), defines an isotropic point of the curve, when t_i is a root of the equation

$$(9)\quad G(t) - iF(t) = 0,$$

or

$$F(t) + iG(t) = 0.$$

Consequently, when the roots $t_1, t_2, \cdots, t_k$ of $r - is = 0$ all define isotropic points, (9) may be written in the form

$$(10)\quad F(t) + iG(t) = (r - is)\Psi(t),$$

where $\Psi(t)$ is a polynomial in t, which in general, has complex coefficients, and which, when u and v denote polynomials in t with real coefficients, may be written in the form

$$\Psi(t) = u + iv.$$

u and v must not vanish identically simultaneously, and may reduce to constants. Now

$$(r - is)(u + iv) = ru + sv + i(rv - su),$$

so that from (10)

$$F(t) = ru + sv, \tag{11}$$

$$G(t) = rv - su.$$

Hence

THEOREM 2: *Every rational circular curve may be parametrically represented in the form*

$$x = \frac{ru + sv}{(r^2 + s^2)\Phi}, \qquad y = \frac{rv - su}{(r^2 + s^2)\Phi}, \tag{12}$$

where all letters on the right hand side stand for polynomials in t, such that r and s have no common roots. The polynomials $ru + sv$, $rv - su$, $(r^2 + s^2)\Phi$, likewise have no common roots. When the degree of $r^2 + s^2$ is $2k$, then the curve has each of the circular points as a k-fold point. Conversely, from (12) *follows readily that every parametric representation as defined by* (12) *represents a circular curve with the circular points as k-fold points.*

When Φ reduces to a constant, which we may place equal to 1, *and the degrees of $ru + sv$ and $rv - su$ are equal to or less than that of $r^2 + s^2$, all infinite points of the curve are at the circular points, so that, in this case, the curve is isotropic.*

3. *Isotropic Curves in a Complex Plane.*

From the parametric representation of an isotropic curve

$$x' = \frac{ru + sv}{r^2 + s^2}, \qquad y' = \frac{rv - su}{r^2 + s^2}, \tag{13}$$

we find

$$x' + iy' = \frac{u + iv}{r + is}. \tag{14}$$

If we replace t in u, v, r, s by the complex variable

$$t = z = x + iy,$$

and put $x' + iy' = z'$, (14) assumes the form

$$z' = f(z)/g(z), \tag{15}$$

in which $f(z) = u + iv$, $g(z) = r + is$, and $f(z)$ is a polynomial whose degree is at most equal to that of $g(z)$, and has no factor in common with $g(z)$. Moreover $g(z)$ has no real roots. Hence

THEOREM 3: *By the rational transformation*

$$z' = f(z)/g(z),$$

in which $f(z)$ and $g(z)$ are polynomials as defined above, the real axis of the z-plane is transformed into a rational isotropic curve. On the other hand, when $f(z)$ and $g(z)$ are arbitrarily given in advance, subject to the conditions that the degree of $f(z)$ is at most equal to that of $g(z)$, that $g(z) = 0$ has no real roots, and $f(z) = 0$ and $g(z) = 0$ have no common roots, then (15) *always defines an isotropic curve.*

The proof of the second part of the theorem follows easily by deriving equations (14) and (13) from (15) in the reversed order.

Instead of separating in $f(z)$ and $g(z)$ the terms with real coefficients from those with complex coefficients, as understood in (14), we can also separate all real from all imaginary terms, when considering $f(z)$ and $g(z)$ as polynomials in x and y. Thus, from (15) we get an expression for z' of the same form as (14), but in which u, v, r, s are now real polynomials in x and y, which satisfy the Riemann-Cauchy differential equations. Formulas (13), with the new meaning of u, v, r, s, represent the same isotropic curve as before, if we let $y = 0$, and if we let $x = t$ assume all values of the real axis.

When in (15) z describes the real axis, the cartesian equation of the corresponding isotropic curve described by z' is obtained in the following manner: For the real axis $z = \bar{z}$; consequently, when

$$\begin{aligned} f(z) &= z^m + a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_m, \\ g(z) &= b_0 z^n + b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n, \end{aligned}$$

with $m \leqq n$, we have the conditional equations,

$$\begin{aligned} &z'(b_0 z^n + b_1 z^{n-1} + b_2 z^{n-2} + \cdots + b_n) \\ &\qquad - (z^m + a_1 z^{m-1} + a_2 z^{m-2} + \cdots + a_m) = 0 \\ &\bar{z}'(\bar{b}_0 z^n + \bar{b}_1 z^{n-1} + \bar{b}_2 z^{n-2} + \cdots + \bar{b}_n) \\ &\qquad - (z^m + \bar{a}_1 z^{m-1} + \bar{a}_2 z^{m-2} + \cdots + \bar{a}_m) = 0. \end{aligned} \tag{16}$$

The resultant of these two equations in z is a polynomial in z' and $\bar{z}'$, which, when set equal to zero, and on replacing z' by $x' + iy'$, $\bar{z}'$ by $x' - iy'$, reduces to the required equation of the isotropic curve. As both equations in (16) are of degree n, the degree of this curve will, in general, be $2n$, which is also apparent from (13). This method of finding the curve described by z', when z describes the real axis, is valid for any rational transformation between z and z', and includes the generation of all rational circular curves.

4. *Rational Circular Cubics.*

As an example for this method, the cartesian equation of all rational circular cubics, with the origin as a singular point, will be derived from the corresponding rational transformation of the superposed complex plane

$$z' = \frac{z^2 + az + b}{(\alpha + i\beta)z + (\gamma + i\delta)}, \tag{17}$$

where $a, b, \alpha, \beta, \gamma, \delta$ are real, and $\alpha\delta - \beta\gamma \neq 0$. Equations (16) have now the form

$$z^2 + [a - (\alpha + i\beta)z']z + b - (\gamma + i\delta)z' = 0, \tag{18}$$

$$z^2 + [a - (\alpha - i\beta)\bar{z}']z + b - (\gamma - i\delta)\bar{z}' = 0. \tag{19}$$

Eliminating z between (18) and (19), the resulting equation in the cartesian plane reduces to

$$\begin{aligned}(\alpha\delta - \beta\gamma)(x'^2 + y'^2)(\beta x' + \alpha y') + (\delta^2 - a\beta\delta + b\beta^2)x'^2 \\ + (2\gamma\delta - a\alpha\delta - a\beta\gamma + 2b\alpha\beta)x'y' \\ + (\delta^2 - a\alpha\gamma + b\alpha^2)y'^2 = 0.\end{aligned} \tag{20}$$

This may represent any rational circular cubic, with the singular point at the origin, by choosing $a, b, \alpha, \beta, \gamma, \delta$ properly.

5. *Rational Bicircular Quartics.*

Instead of treating quartics in the same manner as circular cubics, the reversed order will here be followed. The problem now is, to find the rational transformation in the complex plane, when the equation of the curve is given. The equation of any bicircular rational quartic, with the origin as a singular

point, may be written in the form

$$(21)\quad (x'^2+y'^2)^2+(\alpha x'+\beta y')(x'^2+y'^2) - (p^2+\alpha p)x'^2+bx'y'+cy'^2=0,$$

in which p is the distance from the origin of one of the four points of intersection of the quartic with the x'-axis.

The circle

$$(22)\quad x'^2+y'^2-px'-2ty'=0,$$

through this point and the origin, cuts the quartic in only one variable point, corresponding to the parameter t. The coordinates of (21) can therefore be expressed rationally in terms of t, by solving (21) and (22) simultaneously. The result is

$$(23)\quad x'=\frac{p(4t^2+2\beta t+c)^2-2t(4pt+2\alpha t+\beta p+b)(4t^2+2\beta t+c)}{(4t^2+2\beta t+c)^2+(4pt+2\alpha t+\beta p+b)^2},$$

$$(24)\quad y'=\frac{2t(4pt+2\alpha t+\beta p+b)^2 - p(4t^2+2\beta t+c)(4pt+2\alpha t+\beta p+b)}{(4t^2+2\beta t+c)^2+(4pt+2\alpha t+\beta p+b)^2}.$$

If we put $4t^2+2\beta t+c=r$, $4pt+2\alpha t+\beta p+b=s$, (23) and (24) may be written in the form

$$(25)\quad x'=\frac{ru}{r^2+s^2},$$

$$(26)\quad y'=\frac{-su}{r^2+s^2},$$

which is in agreement with the general result contained in (13), in which $v=0$. Hence

$$z'=\frac{u}{r+is},$$

or, putting $t=z$,

$$(27)\quad z'=\frac{-4(\alpha+p)z^2-2bz+cp}{4z^2+[2\beta+i(4p+2\alpha)]z+i(\beta p+b)}$$

is the required transformation, which transforms the real axis into the given rational bicircular quartic.

For example, Bernoulli's lemniscate

$$(x'^2 + y'^2)^2 - 2a^2(x'^2 - y'^2) = 0 \tag{28}$$

is obtained by the transformation

$$z' = \frac{a\sqrt{2}(z^2 - 1)}{-z^2 - 2iz - 1}. \tag{29}$$

6. *Transformation of a General Algebraic Curve in the z-Plane.*

Instead of restricting ourselves to the rational transformation of the real axis in the z-plane, we may ask the question, what effect the transformation (13) has upon an algebraic curve C of the order μ and of deficiency p, when u, v, r, s are obtained as polynomials of x and y from the rational transformation (15), in which the degree of $g(z)$ is n, that of $f(z)$ m, with $m \leqq n$. When C does not pass through the common intersections of the curves $r = 0$ and $s = 0$, C will intersect the curve $r^2 + s^2 = 0$ in $2\mu n$ imaginary points which by (13) are transformed into the circular points at infinity. Each circular point is therefore a μn-fold point of the transformed curve C'. Now to a straight line l' in the z'-plane corresponds in the z-plane a curve l of order $2n$. This curve l cuts C in $2\mu n$ points, which, conversely, are transformed into the $2\mu n$ intersections of l' with C'. The order of C' is therefore $2\mu n$, and as the circular points absorb all $2\mu n$ infinite points of C', this curve will be an *isotropic curve*. According to Clebsch,* the deficiency of C' is the same as that of C, so that a rational curve C is transformed into a rational isotropic curve C'. Evidently nothing in the generality of representation of rational circular and isotropic curves is lost by taking in place of a general curve C, the real axis. It is therefore not necessary, for our purpose, and in this place, to give further details of the relation between C and C'.

* Vorlesungen über Geometrie, vol. 1, pp. 661–674 (1876).

THE SELF-DUAL PLANE RATIONAL QUINTIC.

BY PROFESSOR L. E. WEAR.

A SELF-DUAL curve is defined to be a curve which has the same number of cusps and double points as it has inflexional tangents and double tangents respectively; and furthermore there are correlations—including polarities—which send the curve into itself.

Haskell, in this BULLETIN, January, 1917, found the maximum number of cusps of an algebraic plane curve, and enumerated the self-dual curves. The well known binomial curves

$$x_1^n = x_0^{n-r}x_2^r$$

have been extensively studied and shown to be self-dual.* The case of the rational plane quartic has been considered in my dissertation at the Johns Hopkins University.†

We here consider briefly the quintic. Since the class of the curve is to equal the order, we have as the fundamental equation,

$$n = n(n-1) - 2d - 3c,$$

where d is the number of double points and c the number of cusps. Hence we have for the quintic,

$$2d + 3c = 15,$$

an equation which has three solutions, as follows:

(1) $d = 0, \quad c = 5,$

(2) $d = 3, \quad c = 3,$

(3) $d = 6, \quad c = 1.$

Case (3) may arise from the degenerate quintic composed of a conic and a cuspidal cubic.

The second case, that of the rational quintic, is the one to be considered here. Furthermore, we consider the curve which

* Loria, Spezielle Ebene Kurven, p. 308; Wieleitner, Algebraische Kurven, p. 136; Snyder, *American Journal*, vol. 30; Winger, *American Journal*, vol. 36.

† "The self-dual plane rational quartic," Dissertation. Johns Hopkins University, May, 1913.

is self-dual in all possible ways, which will be invariant under the largest possible group of transformations. The cusps then will be distinct. By taking the products of the correlations two at a time we obtain the collineations of a collineation group under which the curve is self-projective. These must interchange cusps, say, in all possible ways, and hence the curve must be invariant under a G_6 composed of a cyclic g_3, the elements of which interchange the cusps cyclically, and three elements, obtained by adding to the g_3 an element of period two, and which leave one cusp fixed while interchanging the other two.

Now the equations of the rational quintic invariant under the dihedral G_6 are*

$$(1) \qquad x_0 = t^5 + 5t^2, \qquad x_1 = 5t^3 + 1, \qquad x_2 = t^4 + t.$$

The flexes are $t^3 + 1$, and the cusps $t^3 - 1$. The G_6 is generated by the elements

$$(2) \qquad t' = \omega t, \qquad t' = 1/t, \qquad (\omega^3 = 1),$$

with the appropriate ternary transformations

$$(3) \qquad x_0' = x_0, \qquad x_1' = \omega x_1, \qquad x_2' = \omega^2 x_2.$$

$$(4) \qquad x_0' = x_1, \qquad x_1' = x_0, \qquad x_2' = x_2.$$

Let us now add a correlation which will send any point of the curve into a line of the curve, and vice versa. In particular we desire a correlation that will interchange cusps and flexes, and likewise double points and double lines.

In order to obtain the correlation we need the line equations of the curve, which are obtained by taking the Jacobians of (1) two at a time, and are

$$(5) \quad \xi_0 = 5\tau^3 - 1, \quad \xi_1 = \tau^2(\tau^3 - 5), \quad \xi_2 = -10\tau(\tau^3 - 1).$$

The binary transformation $t\tau = -1$ will send the cusp $t = 1$ into the flex line $\tau = -1$, and conversely. Let us ask that this send *any point* of the curve into a *line* of the curve. Now any point of the curve is

$$(6) \qquad \xi_0(t^5 + 5t^2) + \xi_1(5t^3 + 1) + \xi_2(t^4 + t) = 0$$

and any line is,

$$(7) \qquad x_0(5\tau^3 - 1) + x_1(\tau^3 - 5) - x_2 10\tau(\tau^3 - 1) = 0.$$

* See Winger, l. c., p. 73.

Making the substitution $t = -1/\tau$ in (6), we have, after simplifying,

$$(8) \qquad \xi_0(5\tau^3 - 1) + \xi_1(\tau^5 - 5\tau^2) + \xi_2(\tau - \tau^4) = 0.$$

By identifying (7) with (8) there results the polarity

$$(9) \qquad \xi_0 = x_0, \qquad \xi_1 = x_1, \qquad \xi_2 = 10x_2.$$

Combining (9) with the elements of the collineation group G_6, we obtain altogether six correlations which leave the curve unaltered. The latter statement is true since elements of the G_6 send any point of the curve into a second point, and this transformation followed by (9) must send the original point into a line of the curve. The correlations, with the elements of the G_6, make up a G_{12} of collineations and correlations under which the curve is invariant. The following table gives the elements of the group, binary and ternary:

Collineations.

	x_0'	x_1'	x_2'	t'
1:	$= x_0,$	$= x_1,$	$= x_2,$	$= t.$
S:	$x_0,$	$\omega x_1,$	$\omega^2 x_2,$	$\omega t.$
S^2:	$x_0,$	$\omega^2 x_1,$	$\omega x_2,$	$\omega^2 t.$
T:	$x_1,$	$x_0,$	$x_2,$	$1/t.$
ST:	$x_1,$	$\omega x_0,$	$\omega^2 x_2,$	$\omega/t.$
S^2T:	$x_1,$	$\omega^2 x_0,$	$\omega x_2,$	$\omega^2/t.$

Correlations.

	ξ_0	ξ_1	ξ_2	τ
Π_0:	$= x_0,$	$= x_1,$	$= 10x_2,$	$= -1/t.$
$\Pi_0 S$:	$x_0,$	$\omega x_1,$	$10\omega^2 x_2,$	$-\omega/t.$
$\Pi_0 S^2$:	$x_0,$	$\omega^2 x_1,$	$10\omega x_2,$	$-\omega^2/t.$
$\Pi_0 T$:	$x_1,$	$x_0,$	$10x_2,$	$-t.$
$\Pi_0 \cdot (ST)$:	$x_1,$	$\omega x_0,$	$10\omega^2 x_2,$	$-\omega^2 t.$
$\Pi_0 \cdot (S^2T)$:	$x_1,$	$\omega^2 x_0,$	$10\omega x_2,$	$-\omega t.$

It is easily verifiable that these elements have the group properties. Only the first four of the correlations are polarities, and of these $\Pi_0 T$ alone refers to a real conic, the equation of which is

$$x_0 x_1 + 5x_2^2 = 0.$$

This conic is tangent to the curve at $t = 0$, $t = \infty$, and intersects the curve at six other points. At one of the latter points a tangent to the conic is tangent to the curve at some other point. We may summarize with this theorem: *The self-dual plane rational quintic admitting of the greatest possible number of correlations is invariant under a G_{12} consisting of collineations and correlations.*

THROOP COLLEGE,
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GROUPS CONTAINING A RELATIVELY LARGE NUMBER OF OPERATORS OF ORDER TWO.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society March 29, 1919.)

§ 1. *Introduction.*

IT is well known that every group which contains at least one operator of order 2 must contain an odd number of such operators and that there is an infinite number of groups such that each of them contains exactly $2m + 1$ operators of order 2, where m is an arbitrary positive integer or 0. It is also known that if exactly one half of the operators of a group are of order 2 then the order of this group must be of the form $2(2m + 1)$ and it must be the dihedral or the generalized dihedral group of this order. Moreover, it has been proved that a group G of order

$$g = 2^{\alpha}(2m + 1)$$

cannot contain more than $2^{\alpha}m + 2^{\alpha} - 1$ operators of order 2, α being an arbitrary positive integer, and whenever G contains this number of operators of order 2 it is either the abelian group of order 2^{α} and of type $(1, 1, 1, \cdots)$ or it is the direct product of the abelian group of order $2^{\alpha-1}$ and of type $(1, 1, 1, \cdots)$ and the dihedral or the generalized dihedral group of order $2(2m + 1)$.*

* G. A. Miller, this BULLETIN, vol. 13 (1907), p. 235.

From these theorems it results directly that when the order of a group is given but no other restrictions are imposed on the group, it is always possible to find an integer which represents the upper limit for the number of operators of order 2 contained in a group of this order, and also to state how many groups of this order contain this number of operators of order 2. In particular, there is no group of order 1,000 which contains more than

$$2^3 \cdot 62 + 2^3 - 1 = 503$$

operators of order 2, and there are exactly three groups of order 1,000 which contain separately 503 operators of this order since there are three abelian groups of order p^3, p being a prime number.

When the number of operators of order 2 contained in G exceeds $g/2$, this excess cannot be an even number, for if it were an even number $2m$ it would result that the order of G would have to be twice an odd number. In fact, if we let $2k$ represent the number of the operators of order greater than 2 in such a group, it would follow that

$$2k + 2m + 1 = g/2.$$

Since a group whose order is twice an odd number contains a subgroup of half its order composed of its operators of odd order, it results that $m = 0$ whenever $2m$ represents the number of the operators of order 2 in excess of half the order of the group. That is, *whenever more than half of the operators of a group are of order 2 this excess is an odd number.* This elementary theorem will be generalized in the following section.

Let $g/2 - k$, k being a positive integer, represent the number of the operators of order 2 contained in G. When k is even, g is of the form $2(2m + 1)$ and hence G contains a subgroup of order $2m + 1$. If t represents any operator of order 2 contained in G, the product of t and an operator in this subgroup of order $2m + 1$ cannot be of order 2 unless t transforms this operator into its inverse. As all of the operators of a group must correspond to their inverses whenever more than three-fourths of them correspond to their inverses in an automorphism of the group, it results that when k is even $2m + 1 \leqq 4k$. In particular, *there is only a finite number of groups which satisfy the condition that the number of their operators of order 2 is equal to half the order of the group minus a given even number.*

While the number of the operators of order 2 contained in a group cannot be equal to one half of the order of the group plus a positive even number, it can be equal to one-half the order of the group minus an arbitrary positive number. In fact, there is a cyclic group such that the number of its operators of order 2 is $g/2 - k$, k being an arbitrary positive integer. The order of this cyclic group is clearly $2(k+1)$. When $k=2$ or 4 there is no other group satisfying the given condition, but when $k = 6$ there is also a non-cyclic group of order 18 which involves exactly $9 - 6 = 3$ operators of order 2, as can easily be verified.

§ 2. *Groups in Which More Than One-Half of the Operators are of Order Two.*

Whenever more than one half of the operators of G are of order 2, this excess must be an odd number, as was noted above. We shall now prove that this odd number is always of the form $2^\alpha - 1$. When G is abelian and of type $(1, 1, 1, \cdots)$ it is evident that this condition is satisfied. In all other cases G contains a non-invariant operator s_1 of order 2. Let H_1 represent the subgroup of G composed of all the operators of G which are commutative with s_1 and let $G - H_1$ represent the totality of the operators of G which are not contained in H_1. Since each of the operators of $G - H_1$ is non-commutative with s_1 it results that at least one half of these operators have orders which exceed 2, and hence more than one half of the operators of H_1 are of order 2.

When H_1 is abelian it must be of order 2^n and of type $(1, 1, 1, \cdots)$. If it is non-abelian, it contains a non-invariant operator s_2 of order 2, and we let H_2 represent the subgroup composed of all the operators of H_1 which are commutative with s_2. The totality of operators $H_1 - H_2$ will again contain at least as many operators whose orders exceed 2 as the number of its operators of order 2, and the central of H_2 must exceed that of H_1, which, in turn, exceeds that of G. By continuing this process we must arrive at an abelian group H_m composed of all the operators of H_{m-1} which are commutative with one of its non-invariant operators s_m of order 2. The subgroup H_m has an order of the form 2^n and is of type $(1, 1, 1, \cdots)$.

Since H_m is a subgroup of G it is well known that all the operators of G may be uniquely represented as follows:

$$H_m + H_m t_2 + H_m t_3 + \cdots + H_m t_\gamma.$$

As H_m contains each of the operators $s_1, s_2, \cdots, s_m$, it is evident that at least half of the operators in each of these co-sets have orders which exceed 2. On the other hand, there is at most one of the co-sets $H_mt_2, H_mt_3, \cdots, H_mt_\gamma$ in which the number of operators whose orders exceeds 2 is larger than the number of its operators of order 2. To prove this fact it is only necessary to observe that the number of operators of order 2 in Ht_α, $2 \leqq \alpha \leqq \gamma$, is equal to the order of the subgroup of H_m composed of its operators which are commutative with t_α. If this order were less than 2^{n-1} for two values of α the number of operators of order 2 contained in G would be less than $g/2$, 2^n being the order of H_m.

If the number of operators of order 2 in each of the co-sets $H_mt_2, H_mt_3, \cdots, H_mt_\gamma$ is equal to 2^{n-1} then 2^{n-1} represents also the excess over $g/2$ of the number of the operators of order 2 contained in G. If one of these co-sets Ht_α contains more operators whose orders exceed 2 than operators of order 2 this excess is equal to the number of operators of order 2 which are both contained in H_m and commutative with t_α. Hence it has been proved that *whenever the number of the operators of order 2 contained in a group exceeds one half of the order of the group this excess must be of the form* $2^\alpha - 1$.

From the theorem which has just been proved it is easy to find the form of all the possible ratios between g and the number of operators in G whose orders exceed 2 whenever g is of the form 2^k. In fact, this number is evidently $2^{k-1} - 2^\alpha$ and hence this ratio is always of the form

$$\frac{2^k}{2^{k-1} - 2^\alpha} = \frac{2^\beta}{2^{\beta-1} - 1}.$$

Moreover, there is an infinite system of such groups for every positive integral value of $\beta > 1$.* It may also be noted that whenever one of the co-sets Ht_α, $2 \leqq \alpha \leqq \gamma$, involves more operators whose orders exceed 2 than operators of order 2, this co-set is composed of all the operators of G which are commutative with less than one half of the operators of H_m. Hence this co-set involves the inverses of all its operators and therefore each of its operators transforms H_m into itself. As one of these operators is of order 2 this co-set and H_m generate a group whose order is twice the order of H_m, and hence the order of each operator of this co-set is a divisor of 4.

* G. A. Miller, *Annals of Mathematics*, vol. 7 (1906), p. 57.

We proceed to prove that whenever g is not of the form 2^a then exactly one half of the operators in each of the co-sets Ht_α, $2 \leqq \alpha \leqq \gamma$, are of order 2. If this condition were not satisfied, H_m would involve an operator s_1 which would be commutative with an operator t of odd order contained in G and would transform into its inverse an operator t' of order 4 found in the co-set in which more than half the operators would be of order 4. This follows directly from the fact that all the operators of the co-set involving t are commutative with exactly half the operators of H_m while t' is commutative with at most one fourth of the operators of this subgroup.

Let H_1 be the subgroup composed of all the operators of G which are commutative with s_1. If the product of t' and an operator t_1' of order 2 which is found in H_1 but not in H_m had an order larger than 2 then t_1' and t' would be commutative since this product would be transformed into its inverse by s_1 and hence $(t't_1')^{-1} = t'^{-1}t_1'$. Therefore, it results that t' is transformed either into itself or into its inverse by all the operators of the group generated by the operators of order 2 found in H_1 but not in H_m. These operators clearly generate H_1, since H_1 involves an operator of odd order and this operator must be contained in each of its subgroups of index 2.

Since t could not transform t' into its inverse it follows that t and t' are commutative. Their product must be transformed into its inverse by s_1 and hence we are led to the contradictory equation

$$(t't)^{-1} = tt'^{-1}.$$

As the assumptions that one of the given co-sets contains more operators of order 4 than of order 2 and that the order of G is divisible by an odd prime number led to a contradiction, we have proved that exactly half of the operators of each of these co-sets must be of order 2 whenever g is divisible by an odd prime.

It will now be proved that the subgroup of index 2 under H_m composed of all the operators of H_m which are commutative with t_α is the same for every value of α from 2 to γ. If this were not true, the subgroup formed by the operators of H_m which are commutative with t would involve an operator s_1 which would transform into its inverse an operator t_1 not found in the subgroup H_1 composed of all the operators of G which are commutative with s_1. Just as before, we may

prove that t_1 is commutative with t because it is transformed either into itself or into its inverse by all the operators of order 2 contained in H_1 but not in H_m.

Moreover, tt_1 is transformed into $t_1^{-1}t^{-1} = t^{-1}t_1^{-1}$ whenever the order of this product exceeds 2. If this order were 2, s_1 would transform $(tt_1)^2 = t^2t_1^2$ into $t^2t_1^{-2} = t^2t_1^2$. As this leads to a contradiction and as tt_1 is also transformed into tt_1^{-1} by s_1, it results that the two assumptions that G contains operators of odd order and that some of the operators of H_m which are commutative with a certain t_α are not commutative with every t_α, $2 \leqq \alpha \leqq \gamma$, are contradictory. It therefore results that *exactly half of the operators of H_m constitute the central of G whenever g is divisible by an odd prime number.*

Let K represent the central of G and suppose that g is divisible by an odd prime number. The quotient group G/K has an order which is divisible by all the odd divisors of g and at least one half of its operators are of order 2. If exactly half of these operators are of order 2, this quotient group is either the dihedral or the generalized dihedral group whose order is of the form $2(2m + 1)$. If more than one half of its operators are of order 2, we may proceed as above and find a second quotient group in which at least one half of the operators are of order 2. Hence we have established the following theorem: *If the order of a group is $2^\alpha(2m + 1)$, $m > 0$, and if more than one half of its operators are of order 2 then this group contains an invariant subgroup of order $2^{\alpha-1}$ and the corresponding quotient group is either the dihedral or the generalized dihedral group of order $2(2m + 1)$.*

The subgroup of G which corresponds to the subgroup of order $2m + 1$ in the quotient group does not involve any of the operators of H_m which transform each of the operators of G whose orders exceed 2 into their inverses, since more than one half of the operators of the former subgroup have orders greater than 2. This subgroup must be abelian since all of its operators whose orders exceed 2 correspond to their inverses in an automorphism and the products of these operators must also correspond to their inverses. It therefore results from the preceding theorem that, *if a group whose order is divisible by an odd prime number has the property that at least one half of its operators are of order 2, it is either a dihedral or a generalized dihedral group.* It also results that H_m is identical with H_1 whenever g has an odd prime factor but this is not necessarily true when g is of the form 2^α, as can easily be verified.

THE DERIVATIVE OF A FUNCTIONAL.

BY PROFESSOR P. J. DANIELL.

In his book on Integral Equations* Volterra has given a definition of the derivative of a functional and has stated somewhat restricted conditions under which the variation can be expressed as a linear integral. In the present paper it is shown that, under more general conditions, the variation is a linear functional in the sense of Riesz* and, therefore, a Stieltjes integral. This theorem is assumed as a condition in a paper by Fréchet.† Let

$$F[f(x)]_a^b$$

denote a functional F of a continuous function $f(x)$ $(a \leqq x \leqq b)$. With Volterra we shall consider only continuous functions. Let us denote the first variation by

$$D(f\,;\,\varphi) = \lim_{\epsilon=0} \frac{1}{\epsilon}\{F[f+\epsilon\varphi] - F[f]\}.$$

In place of Volterra's four conditions we take the two following:

I. $F[f]$ satisfies the Cauchy-Lipschitz condition, namely that we can find a number M such that

$$|F[f_1] - F[f_2]| \leqq M \max|f_1(x) - f_2(x)|.$$

II. The first variation $D(f'\,;\,\varphi)$ exists for all continuous φ, and all continuous f' in the neighborhood of f; that is to say that a number $\eta > 0$ can be found so that the variation exists so long as

$$\max|f'(x) - f(x)| \leqq \eta.$$

Under these conditions the variation is a linear functional, and therefore a Stieltjes integral,

$$D(f\,;\,\varphi) = \int_a^b \varphi(x)d\alpha(x).$$

* V. Volterra, Equations Intégrales, p. 12 et seq. F. Riesz, *Annales de l'Ecole Normale Supérieure*, vol. 31 (1914), p. 9.

† M. Fréchet, *Transactions Amer. Math. Society*, vol. 15 (1914), p. 135.

In the first place

$$D(f\,;\,l\varphi) = \lim_{\epsilon=0} \frac{l}{l\epsilon}\{F[f+\epsilon l\varphi] - F[f]\} \tag{1}$$
$$= lD(f\,;\,\varphi).$$

If we choose $\epsilon > 0$ so small that

$$\epsilon|l|\max|\varphi_1| + \epsilon|m|\max|\varphi_2| < \eta,$$

$D(f+\epsilon m\varphi_2\,;\,\varphi_1)$ will exist by II, and

$$F[f+\epsilon l\varphi_1+\epsilon m\varphi_2] - F[f+\epsilon m\varphi_2] = \epsilon l D(f+\epsilon m\varphi_2;\,\varphi_1) + \epsilon\delta,$$
$$F[f+\epsilon l\varphi_1] - F[f] = \epsilon l D(f\,;\,\varphi_1) + \epsilon\delta',$$

where δ, δ' approach 0 with ϵ. Then

$$P(l\varphi_1, m\varphi_2) = \lim_{\epsilon=0} \frac{1}{\epsilon}\{F[f+\epsilon l\varphi_1+\epsilon m\varphi_2] - F[f+\epsilon m\varphi_2] \tag{2}$$
$$- F[f+\epsilon l\varphi_1] + F[f]\}$$
$$= l\lim_{\epsilon=0}[D(f+\epsilon m\varphi_2\,;\,\varphi_1) - D(f\,;\,\varphi_1)].$$

Similarly

$$P(l\varphi_1, m\varphi_2) = m\lim_{\epsilon=0}[D(f+\epsilon l\varphi_1\,;\,\varphi_2) - D(f\,;\,\varphi_2)]. \tag{3}$$

The expression (2) is the product of l and a function of m independent of l, while (3) is the product of m and a function of l only, and they are equal. Each must be a product of lm and an expression K independent of l, m.

$$P(l\varphi_1, m\varphi_2) = lmK(\varphi_1, \varphi_2).$$

In this make $l = 1 = m$; then

$$P(\varphi_1, \varphi_2) = K(\varphi_1, \varphi_2).$$

Or

$$P(l\varphi_1, m\varphi_2) = lmP(\varphi_1, \varphi_2).$$

Making $m = l$,

$$P(l\varphi_1, l\varphi_2) = l^2P(\varphi_1, \varphi_2).$$

But

$$P(l\varphi_1, l\varphi_2) = l\lim_{\epsilon=0}\frac{1}{\epsilon l}\{F[f+\epsilon l\varphi_1+\epsilon l\varphi_2]$$
$$- F[f+\epsilon l\varphi_2] - F[f+\epsilon l\varphi_1] + F[f]\}$$
$$= lP(\varphi_1, \varphi_2).$$
$$\therefore\ l^2P(\varphi_1, \varphi_2) = lP(\varphi_1, \varphi_2)$$
$$P(\varphi_1, \varphi_2) = 0.$$

Or

$$\lim_{\epsilon=0} \frac{1}{\epsilon} \{F[f+\epsilon\varphi_1+\epsilon\varphi_2] - F[f] - F[f+\epsilon\varphi_2] + F[f] - F[f+\epsilon\varphi_1] + F[f]\} = 0,$$

$$D(f;\varphi_1+\varphi_2) - D(f;\varphi_2) - D(f;\varphi_1) = 0. \tag{4}$$

Combining (1) and (4), we see that

$$D(f;c_1\varphi_1+c_2\varphi_2) = c_1D(f;\varphi_1) + c_2D(f;\varphi_2).$$

Thus the variation is distributive in φ. Secondly, from condition I,

$$|F[f+\epsilon\varphi] - F[f]| \leqq M\epsilon \max|\varphi|,$$

or

$$|D(f;\varphi)| = \lim_{\epsilon=0} \frac{1}{\epsilon}|F[f+\epsilon\varphi] - F[f]| \leqq M \max|\varphi|.$$

The variation is also bounded, considered as an operation on φ. This proves it to be a linear functional by Riesz's definition. To find the integrating function $\alpha(x)$ we may proceed as follows:

Let $\varphi(x; c, d)$ denote the continuous function,

$$\varphi = 1, \quad a \leqq x \leqq c,$$
$$= 0, \quad d \leqq x \leqq b,$$
$$\varphi \text{ linear from } c \text{ to } d.$$

Then

$$\alpha(c) = \lim_{d=c} D(f;\varphi),$$

and in general for any continuous $\varphi(x)$,

$$D(f;\varphi) = \int_a^b \varphi(x)d\alpha(x).$$

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SCRITTI MATEMATICI.

Scritti matematici offerti ad Enrico D'Ovidio in occasione del suo LXXV genetliaco, 11 Agosto 1918, e publicati per cura di FRANCESCO GERBALDI *e* GINO LORIA. Torino, Fratelli Bocca, editori, 1918. 8vo, 386 pages. Lire 30.

THE title of this volume sufficiently indicates its general character. A beautiful portrait forms the frontispiece, representing Professor D'Ovidio as he will be remembered by many hundreds of Italians as well as by a number of Americans who, like the writer, have had the good fortune, at one time or another, to be among those in attendance at the University of Turin.

There is a short preface written by Professors Gerbaldi and Loria in which is set forth the raison d'être for the volume. To quote the opening sentence—"On the approach of the day on which an inflexible law would retire Senator Enrico D'Ovidio from the university chair, there arose in the minds of many students whom he has had in his long and glorious career as a teacher, the pleasant idea of choosing this occasion —which coincides with his 75th birthday—to manifest to him their sentiments of unalterable affection and, at the same time, to present to him their sincere good wishes ad multos annos."

Of the 103 persons contributing to the expense of publication, and whose names appear directly following the preface, 47 have been actual students under Professor D'Ovidio, and 20 have been, at one time or another, directly associated with him as Assistenti in the University of Turin. The list contains many names well known to students of mathematics the world over, and bears witness to the great influence Professor D'Ovidio has had upon the growth of mathematics and upon the teaching of mathematics, not alone in Italy, but, mainly perhaps through his disciples, throughout the civilized world. This influence has been exerted from the University of Turin for more than forty years.

Following the list of contributors is a catalogue of 91 publications by Professor D'Ovidio. Some of these are problems or solutions of problems; two or three are elementary texts; a number are commemorative addresses or biographical

papers; the majority deal with problems in geometry; several, the most important perhaps, are concerned with the theory of binary forms. A glance at the titles of these papers leads to the conviction that Professor D'Ovidio's great influence has come about rather more through personal contact as a teacher than through published writings, and goes to show that there is such a thing as creative teaching as well as there is creative scholarship—a fact worth noting in a time when so much emphasis is placed upon the latter function and apparently so little is thought about the former.

It would be out of place here to review in detail the twenty-one papers forming the body of the volume. The titles and authors of these papers with perhaps a word or two indicating the content must suffice.

I. "Su alcune classi particulari di sistemi continui di quadriche, e sui rispettivi inviluppi." CORRADO SEGRE, Turin.

The lines of ordinary 3-space can be imaged as points upon a V_4^2 in S_5. A study of point loci upon the V_4^2 leads to properties of ruled surfaces in S_3. In particular, sections of the V_4^2 by the planes in S_5 are images of quadratic reguli in S_3. Professor Segre is thus led to what he calls concatenated quadric surfaces and quartic curves of the first kind, and to systems of concatenated quadrics and quartics. A quadric surface and a quartic curve of the first kind lying on the surface are concatenated, or chained together, if a simple quadrilateral inscribed in the latter has its sides lying upon the former. The present paper is evidently an outgrowth of a recent paper by Professor Segre in *Annali di Matematica* (3) 27, (1918), page 151.

II. "Le frazione continue di Halphen." FRANCESCO GERBALDI, Pavia.

The continued fractions of Halphen (Traité des Fonctions elliptiques, second volume) arise from the expression $\frac{\sqrt{X}-\sqrt{Y}}{x-y}$, where X and Y are the values of a biquadratic expression when the variable assumes the values x and y, respectively. Halphen, and Jacobi before him, made use of elliptic functions in arriving at the development of irrational expressions like the above in continued fractions. Professor Gerbaldi returns to the more elementary methods of Abel in the present paper.

Two other papers on the same subject have appeared;—one in Volume 53 of *Atti di Torino*, and the other in Volume 51 of *Rendiconti R. I. Lombardo*.

III. "Le cubiche gobbe aventi ciascuno all'infinito tre punti reali e distinti." GINO LORIA, Genoa.

In 1879 D'Ovidio published an exhaustive study of twisted cubics making use of the symbolic notation of binary forms (*Memorie R. Accademia Torino* and *Giornale di Matematiche*). Professor Loria here studies the cubic hyperbolas, using only elementary means to arrive at the properties of these curves.

IV. "Intorno ad un tipo notevoli di sistemi lineari di reciprocità degeneri tra spazi ad n dimensioni." EUGENIO TOGLIATTI, Turin.

The study of projective correspondences between hyperspaces leads to the consideration of types of degenerate transformations (cf. Bertini; Introduzione alla Geometria proiettiva degli Iperspazi). The present paper follows and is an extension of a note by the same author published in *Atti di Torino*, 1916–17.

V. "Sulle congruenze W di cui una falda focale e una quadrica." ALESSANDRO TERRACINI, Turin.

One may consider a correspondence set up in which homologous points are the foci on the same ray of a congruence. If, in this correspondence, asymptotic lines on the focal mantels of the congruence are homologous lines, the congruence is called a W-congruence. The author here studies the W-congruences in which one of the focal mantels is a quadric.

VI. "Alcune osservazioni relative ai problemi secondarii della balistica esterna." GUIDO FUBINI, Turin.

A study of the effect of the wind and of the rotation of the earth upon projectiles, with the hope of modifying the classical differential equations involved so that theoretical results will be in closer accord with experimental data.

VII. "Sulle curve che posseggono una infinità continua di correspondenze algebriche." GUIDO CASTELNUOVO, Rome.

A curve of genus greater than unity does not admit of an infinity of birational transformations into itself (Schwarz,

Crelle, 1875). Castelnuovo here studies those curves each of which is transformable into itself by a continuous infinity of non-birational algebraic correspondences and arrives at a generalization of the Schwarz theorem. The paper is based upon the classic memoir of Hurwitz (*Mathematische Annalen*, 1886).

VIII. "Le oscillazione armoniche nelle antenne radiotelegrafiche direttamente eccitate." LUIGI LOMBARDI, Naples.

This paper contains results of some experiments with radio apparatus at the polytechnic school in Naples.

IX. "Sugli integrali semplici di 1ª specie appartenenti ad una superficie algebrica." FRANCESCO SEVERI, Padua.

A development of one of Severi's brief notes communicated to the Académie des Sciences, Paris, in 1911 (*Comptes Rendus*, volume 152, page 1079).

X. "Sopra alcune applicazioni della teoria dell'urto." EMILIO ALMANSI, Rome.

A brief study of the theory of impact in a system of moving bodies with application, in particular, to three spheres.

XI. "Generalizzazione di una trasformazione di d'Ocagne." ANGELO PENSA, Turin.

The transformations of d'Ocagne may be found explained in an article by M. d'Ocagne, *American Journal*, volume 11 (1889), pages 55–70. Pensa here generalizes the problems raised by d'Ocagne (solved in particular cases by himself and others) and reaches general results by means of vector analysis.

XII. "Estensione e studio di un metodo di sommazione generico di Borel." GUSTAVO SANNIA, Cagliari.

This paper follows one by the same author published in *Rendiconti del Circolo Matematico di Palermo*, volume 42, 1917. Both papers are based upon Borel's summation formula (Leçons sur les Séries divergentes, Gauthier-Villars, 1901).

XIII. "Sopra la propagazione di onde in un mezzo indefinito." ERNESTO LAURA, Pavia.

This paper deals with the motion of waves in a fluid medium when, in particular, the waves are reflected from a fixed surface. It follows an article by the same author published in *Atti di Torino*, 1915.

XIV. "Problemi sulla determinazione delle linee sghembe." MATTEO BOTTASSO,* Messina.

This paper deals with certain generalizations of the transformation of Combescure (Bianchi, Lezione di Geometria Differenziale, second edition, page 40). It follows a note by Bottasso published in *Atti di Torino*, 1917–18.

XV. "Riflessioni sopra alcuni principii della teoria degli aggregati e della funzioni." BEPPO LEVI, Parma.

A logico-mathematical discussion dealing in particular with the Zermelo postulate (*Mathematische Annalen*, volume 59).

XVI. "Nuovo metodo per la risoluzione diretta dell'equazione $ax + by = c$ in numeri interi e positivi, quando i tre numeri noti a, b, c sono interi e positivi." GIUSEPPE BERNARDI, Bologna.

A short paper whose title sufficiently indicates its content.

XVII. "Un interessante problema di geodesia pratica." NICODEMO JADANZA, Turin.

A discussion of the problem: To determine the mean error of two sides of a geodetic triangle when the third side and the three angles are measured.

XVIII. "Resto nelle formule di interpolazione." GIUSEPPE PEANO, Turin.

A proof of an interesting theorem in determinants from which results, in particular, the well known Lagrange interpolation formula together with a remainder.

XIX. "Questioni elementari di massimo e minimo." FILIBERTO CASTELLANO, Turin.

Some problems in maxima and minima treated by algebraic equations and inequalities.

XX. "Sulle varietà algebriche a tre dimensioni a superficie-sezioni razionali." GINO FANO, Turin.

An abstract from a memoir with the same title published in *Annali di Matematica*, (3), volume 24 (1915).

XXI. "Introduzione alla teoria della forme in più serie di variabili." GIOVANNI GIAMBELLI, Messina.

A generalization of the theory of connexes (Clebsch-

* Died Oct. 3, 1918.

Lindemann). The paper follows two others on the same subject published in Turin by the author in 1910 and 1912.

One cannot turn aside from reading these papers, however cursorily, without a feeling of profound admiration for the scholars who have made the book possible, and especially at at a time when the strain of the war was still in force. Each paper is a distinct contribution to knowledge or else a fuller development of such a contribution recently commenced. The total is a wholly worthy epitome of scientific activity even in normal times. It must, indeed, be a source of great satisfaction to Professor D'Ovidio to have so distinct a proof of the esteem with which his many students, associates, and friends regard his long service and his personal qualifications as an inspiring teacher. One cannot do better, in this connection, than to quote again from the preface. "And we are certain that to the loved teacher our publication will be doubly gratifying in as much as it serves also to show how Italy, in the tragic hours in which we live—not less than in the more grave and decisive periods of her earlier struggles for redemption—has not ceased to feed the sacred flame of science."

L. WAYLAND DOWLING.

SHORTER NOTICES.

Plane Geometry. By E. LONG and W. C. BRENKE. New York, The Century Company, 1916.

THIS text has several very good features. First, before a theorem is demonstrated a method of attack is given. Second, frequent use is made of algebra and thereby many blind proofs are avoided. Third, construction work is introduced early in the course. Fourth, areas are introduced before proportion. Fifth, a little trigonometry and analytics is given.

The main fault with the book is that it contains quite a number of inaccurate statements, e. g., "Place the triangles with their longest sides together," page 32; page 107, $c^2 = h^2 + (b - a)^2$ is only true if A is acute and that is not at all necessary; page 204, the definitions of the trigonometric functions are incorrect.

F. M. MORGAN.

Plane Geometry. By JOHN H. WILLIAMS and KENNETH P. WILLIAMS. Chicago, Lyons and Carnahan, 1915.

THIS text in my estimation is an excellent example of what a plane geometry book should not be. It starts with a lot of formal definitions followed by demonstrative work. There are so many inaccurate and incorrect statements, definitions, and proofs that I think it undesirable to list them. I shall however state a few: "An exterior angle of a triangle is the angle formed by producing one side of the triangle"; "A triangle is defined as a polygon bounded by three straight lines," and twenty pages later we find the definition of a polygon. "The limit of a variable is a constant which the variable is supposed to approach in value and can be made to differ from it by an amount that is less than any assignable value, but can not be made absolutely equal to it."

F. M. MORGAN.

Theories of Energy. By HORACE PERRY. New York, G. P. Putnam's Sons, 1918. vii + 231 pp.

THE title of this book is as misleading as the results are unsatisfactory. The author does not consider "theories of energy," but advances a theory of energy due to his own reading and reflection. The plural character is due to the fact that he considers as theories: theory of energial propagation, theory of the energetic atom, theory of spectral lines, theory of gravity, theory of color, etc. The unsatisfactory character lies in both the results of his reflections and the gaps in his theory.

In the first place (and to endeavor to point out the very long entire list of features of his explanation of energy would be tedious and unprofitable) we need to notice his idea of matter. He begins on the first page with the assertion that "All space is filled with matter, and in the infiniteness of space there is no vacuity anywhere, not even of the extent of an atom's size, and the universe, embracing all the matter in existence, is continuous throughout." This idea that the entirety of space is filled with a continuous material medium was new when Thomson's vortex atom was at its best, but is far in the past at present. Perry's ether is perfectly continuous in all space, but as it has no cohesion between its parts, there is "merely a togetherness without any forcible hold." It is a "perfect fluid with perfect passability."

This ether is subject to condensations and rarefactions, which are propagated in all directions. The character of the wave thus produced seems not to enter his calculations.

Matter is continuous. In fact the electron is not anywhere mentioned in the theory he advances. It is endowed with energy in the form of heat and chemical energy. The radioactive disintegration is merely the chipping at the surface of the atom due to the motion of the internal energy. Energy is then defined to be the internal action of the atom, all energy being *densitic* in character, which means it consists of condensation or rarefaction waves. The wave frequency differentiates the various forms of energy. Gravity on this scheme is of a single frequency.

Magnetism is energy generated in the centers of the iron atoms, being "the natural energy of the iron, modified by the substance with which the iron is combined in molecules, and augmented through a certain method of reciprocal energizement." The remaining definitions may be passed over. The author seems to be familiar with a number of antiquated textbooks of a college grade, and somewhat with modern phenomena. From these he has attempted to build up an explanation of the very intricate laws and phenomena of the whole of physics. The result is what would have been expected.

JAMES BYRNIE SHAW.

NOTES.

THE April number (volume 20, number 2) of the *Transactions of the American Mathematical Society* contains the following papers: "Memoir on the general theory of surfaces and rectilinear congruences," by G. M. GREEN; "Modular concomitant scales, with a fundamental system of formal covariants, modulo 3, of the binary quadratic," by O. E. GLENN; "Concerning a set of postulates for plane analysis situs," by R. L. MOORE; "On the limit functions of sequences of continuous functions converging relatively uniformly," by E. W. CHITTENDEN.

AT the meeting of the National Academy of Sciences held at Washington April 28–30, Professors OSWALD VEBLEN, E. J.

WILCZYNSKI, and E. B. WILSON were elected members. Professor EDWARD KASNER read a paper on "Geometry of the wave equation."

AT the meeting of the Edinburgh mathematical society on May 9, the following papers were read: By W. H. METZLER, "Theorems concerning the differentiation of a circulant;" by T. M. MACROBERT, "The integrals of the hypergeometric equation."

THE Indian mathematical society held its second conference at Bombay, January 11–13, in connection with the session of the Indian science congress. Professor A. C. L. WILKINSON, of Elphinstone College, Bombay, is president. The society has a membership of 195.

THE following university courses in mathematics are announced:

COLUMBIA UNIVERSITY (academic year 1919–1920).—By Professor T. S. FISKE: Differential equations, four hours.—By Professor F. N. COLE: Invariants and higher plane curves, three hours (second term); Theory of groups, three hours.—By Professor JAMES MACLAY: Theory of functions, four hours (first term).—By Professor D. E. SMITH: History of mathematics, two hours; Practicum in the history of mathematics, four hours.—By Professor C. J. KEYSER: Modern theories in geometry, four hours.—By Professor EDWARD KASNER: Seminar in differential geometry, two hours.—By Professor W. B. FITE: Differential equations, three hours.—By Dr. J. F. RITT: Transcendental functions, three hours (second term).

CORNELL UNIVERSITY (academic year 1919–1920).—By Professor JAMES MCMAHON: Mathematics of insurance and probabilities; Actuarial science.—By Professor J. H. TANNER: Mathematics of finance.—By Professor VIRGIL SNYDER: Birational transformations; Theory of equations (second term).—By Professor F. R. SHARPE: Fourier series and the potential function.—By Professor ARTHUR RANUM: Non-euclidean geometry (first term); Theory of numbers (second term).—By Professor W. B. CARVER: Projective geometry.—By Professor D. C. GILLESPIE: Elementary differential equations;

Calculus of variations.—By Professor W. A. HURWITZ: Theory of groups; Vector analysis.—By Professor C. F. CRAIG: Theory of differential equations; Mathematics for teachers. —By Professor F. W. OWENS: Advanced calculus; Mechanics. —By Dr. H. B. OWENS: Advanced analytic geometry.

All courses are three hours a week.

HARVARD UNIVERSITY (academic year 1919–1920).—All courses meet three times a week throughout the year, except those marked *, which meet for half a year.—By Professor W. F. OSGOOD: Differential and integral calculus (advanced course); Infinite series and products;* Galois's theory of equations.*—By Professor C. L. BOUTON: The elementary theory of differential equations;* Differential equations, with an introduction to Lie's theory of continuous groups.—By Professor J. L. COOLIDGE: Introduction to modern geometry and modern algebra; Projective geometry;* Non-euclidean geometry.*—By Professor E. V. HUNTINGTON: The fundamental concepts of mathematics.*—By Professor O. D. KELLOGG (of the University of Missouri): Introduction to the theory of potential functions and Laplace's equation;* Vector analysis.*—By Professor G. D. BIRKHOFF: Differential and integral calculus (advanced course);* The analytical theory of heat and problems in elastic vibrations;* The partial differential equations of mathematical physics.—By Professor —— ——: The theory of functions (introductory course); Developments in series;* Dynamics (second course);* By Professor W. C. GRAUSTEIN: Differential geometry of curves and surfaces.—By Dr. H. C. M. MORSE: Elliptic functions;* Automorphic functions.*—By Dr. I. A. BARNETT: Integral equations;* Functions of lines.*

Professor KELLOGG will conduct a fortnightly seminar in analysis. Courses of research are also offered by Professor OSGOOD in the theory of functions, by Professor BOUTON in the theory of point transformations, by Professor COOLIDGE in geometry, by Professor KELLOGG in analysis, by Professor BIRKHOFF in the theory of differential equations, and by Professor GRAUSTEIN in geometry.

PRINCETON UNIVERSITY (academic year 1919–1920).—By Professor H. B. FINE: Functions of a complex variable.—By Professor L. P. EISENHART: Differential geometry.—By

Professor OSWALD VEBLEN: Seminar in analysis situs.—By Professor PIERRE BOUTROUX: Linear differential equations.—By Professor J. H. M. WEDDERBURN: Higher algebra.

All courses are three hours a week.

UNIVERSITY OF CHICAGO. Autumn quarter:—By Professor E. H. MOORE: Seminar in general analysis, I, two hours; Matrices in general analysis, four hours.—By Professor G. A. BLISS: Theory of functions of a real variable, four hours; Calculus, I, five hours.—By Professor L. E. DICKSON: Continuous groups, four hours; Elementary theory of equations, four hours.—By Professor H. E. SLAUGHT: Differential equations, four hours.—By Professor A. C. LUNN: Vector analysis, four hours; Applied mathematics, five hours. Winter quarter:—By Professor E. H. MOORE: Seminar in general analysis, II, two hours; Theory of functions of infinitely many variables in general analysis, four hours.—By Professor G. A. BLISS: Calculus of variations, five hours; Calculus, II, five hours.—By Professor L. E. DICKSON: Theory of algebraic invariants, four hours; Solid analytics, four hours.—By Professor H. E. SLAUGHT: Theory of definite integrals, four hours.—By Professor E. J. WILCZYNSKI: Projective geometry, I, four hours.—By Professor A. C. LUNN: Applications of vector analysis in the theory of electromagnetism, four hours; Applied mathematics, II, five hours.—By Professor J. W. A. YOUNG: Calculus, I, five hours. Spring quarter:—By Professor E. H. MOORE: Seminar in general analysis, III, two hours; Theory of functions of infinitely many variables in general analysis, II, four hours.—By Professor G. A. BLISS: Functions of lines, four hours; Calculus, III, five hours.—By Professor L. E. DICKSON: Finite groups, four hours.—By Professor E. J. WILCZYNSKI: Higher geometry, four hours; Projective geometry, II, four hours.—By Professor A. C. LUNN: The theory of relativity, four hours; Applied mathematics, III, five hours.—By Professor J. W. A. YOUNG; Limits and series, four hours; Calculus, II, five hours.

UNIVERSITY OF ILLINOIS (academic year 1919–1920).—By Professor E. J. TOWNSEND: Differential equations and advanced calculus; Functions of real variables.—By Professor G. A. MILLER: Theory of equations (first term); Theory of groups, II.—By Professor J. B. SHAW: Fundamental functions

(first term); Functional transformations (second term).—By Professor A. B. COBLE: Automorphic functions; Solid analytic geometry (second term).—By Professor R. D. CARMICHAEL: Linear difference equations.—By Professor ARNOLD EMCH: Algebraic surfaces.—By Professor A. R. CRATHORNE: Actuarial theory.—By Professor A. J. KEMPNER: Modern algebra.—By Dr. E. B. LYTLE: History of mathematics, two hours (second term).—By Dr. HENRY BLUMBERG: Projective geometry. All courses are three hours a week unless otherwise designated.

UNIVERSITY OF PENNSYLVANIA (summer session, July 8 to August 16).—By Professor G. H. HALLETT: Higher calculus, five hours.—By Professor H. H. MITCHELL: Mathematical theory of probability, five hours.—By Professor R. L. MOORE: Introduction to the theory of functions of a complex variable, five hours.

YALE UNIVERSITY (academic year 1919–1920).—By Professor JAMES PIERPONT: Elliptic functions.—By Professor P. F. SMITH: Foundations of geometry.—By Professor E. W. BROWN: Hydrodynamics with applications to aeronautics.—By Professor W. R. LONGLEY: Theory of differential equations.—By Professor J. I. TRACEY: Modern geometry, including differential geometry (a first course).—By Mr. J. K. WHITTEMORE: Differential geometry.—By Dr. J. R. KLINE: Advanced algebra. All courses are two hours a week.

THE announcement in the May BULLETIN of the death of Professor H. G. ZEUTHEN was an error which the responsible editor is more than glad to correct. Professor Zeuthen celebrated on February 15 his eightieth birthday. The BULLETIN wishes this Nestor of Danish mathematicians many happy returns.

AT Harvard University, assistant professor G. D. BIRKHOFF has been promoted to a full professorship of mathematics. Assistant professor W. C. GRAUSTEIN, of Rice Institute, has been appointed assistant professor, and Mr. B. H. BROWN and Mr. C. A. RUPP, JR., have been appointed instructors. Professor O. D. KELLOGG, on leave of absence from the University of Missouri, has been appointed lecturer for the academic year 1919–1920.

IN the department of mathematics of the Naval academy at Annapolis, the following promotions have been announced: to be professor, Mr. PAUL CAPRON; to be associate professors, Mr. W. J. KING and Mr. J. B. EPPES; to be assistant professors, Mr. J. A. BULLARD, Mr. JOHN TYLER, Mr. ARTHUR KIERNAN, Mr. J. N. GALLOWAY, Mr. ALEXANDER DILLINGHAM, and Dr. G. R. CLEMENTS. Dr. G. H. CRESSE, of the University of Michigan, has been appointed instructor.

AT the University of Minnesota, Professor G. N. BAUER remains on leave of absence until January 1, 1920. Assistant professor DUNHAM JACKSON, of Harvard University, has been appointed professor of mathematics, and Major W. L. HART has been appointed assistant professor. Mr. R. W. BRINK has been promoted from an instructorship to an assistant professorship, and granted leave of absence for the academic year 1919–1920, during which time he will be lecturer at the University of Edinburgh. Miss MINNA J. SCHICK has been appointed instructor in mathematics.

DR. C. E. WILDER has been appointed assistant professor of mathematics at Northwestern University.

MR. P. A. FRALEIGH, of Cornell Uuniversity, has been appointed instructor in mathematics at Dartmouth College.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

ANDERSEN (A. F.), BOHR (H.) og MOLLERUP (J.). Nyere Undersøgelser over Integralregningen. Kjøbenhavn, 1917. 63 pp.

BOHR (H.). See ANDERSEN (A. F.).

BOREL (E.). Die Elemente der Mathematik. Vom Verfasser genehmigte deutsche Ausgabe von P. Stäckel. Band I: Arithmetik und Algebra nebst den Elementen der Differential-Rechnung. 2te Auflage. Leipzig, Teubner, 1919. 8vo. 16 + 404 pp. M. 12.00

CASTELNUOVO (G.). Calcolo delle probabilità. Milano, Albrighi, Segati e compagnia, 1919. 23 + 373 pp. L. 20.00

HALPHEN (G. H.). Œuvres publiées par les soins de C. Jordan, H. Poincaré, E. Picard, avec la collaboration de E. Vessiot. Tome 2. Paris, Gauthier-Villars, 1918. 7 + 560 pp. Fr. 40.00

HJELMSLEV (J.). Lærebog i Gemoetri. Til Brug ved den Polytekniske Læreanstalt. 1ste Afsnit. Kjøbenhavn, Gjellerup, 1917. 8vo. 208 pp.

JORDAN (C.). See HALPHEN (G. H.).

MILNE (W. P.) and WESTCOTT (J. B.). A first course in the calculus. Part 1: Powers of x. London, Bell, 1918. 20 + 196 pp. 3s. 6d.

MOLLERUP (J.). See ANDERSEN (A. F.).

PASCAL (E.). Lezioni di calcolo infinitesimale. 2a parte: Calcolo integrale. 4a edizione riveduta. (Manuali Hoepli.) Milano, Hoepli, 1919. 24mo. 8 + 330 pp. L. 4.50

PICARD (E.). See HALPHEN (G. H.).

POINCARÉ (H.). See HALPHEN (G. H.).

RICCI-CURBASTRO (G.). Lezioni di analisi algebrica ed infinitesimale. Padova, tip. Universitaria, 1918. 8vo. 13 + 645 pp. L. 27.00

STÄCKEL (P.). See BOREL (E.).

STUTZ (A.). Wahrscheinlichkeitsrechnung. Zürich, Leemann, 1916. 4to. 27 pp.

VESSIOT (E.). See HALPHEN (G. H.).

WESTCOTT (J. B.). See MILNE (W. P.).

WESTIN (O. E.). Grafisk räkning med komplexa tal. Stockholm, Norstedt, 1918. 8vo. 67 pp. Kr. 2.50

WIELEITNER (H.). Algebraische Kurven. II: Allgemeine Eigenschaften. (Sammlung Göschen, Nr. 436.) Berlin, Göschen, 1918. M. 1.25

II. ELEMENTARY MATHEMATICS.

AHRENS (W.). Hebräische Amulette mit magischen Zahlenquadraten. Berlin, L. Lamm, 1916. 8vo. 19 pp. M. 1.00

ALLEN (F.). See TAYLOR (E. H.).

CRANTZ (P.). Arithmetik und Algebra zum Selbstunterricht, I und II. (Aus Natur und Geisteswelt, Nr. 120 und Nr. 205.) 5te und 4te Auflage. Leipzig, Teubner, 1918. 16mo. 114 + 110 pp. Geb. M. 1.50 + M + 1.50

CROCKETT (C. W.). Mathematical review. Arithmetic, algebra, geometry. Troy, Rensselaer Polytechnic Institute, 1918. 12mo. 110 pp. $0.75

GALLUCCI (G.). Algebra e geometria metrica per il 2o corso liceale classico e moderno. Milano, Albrighi, Segadi e Compagnia, 1918. L. 3.00

HJELMSLEV (J.). Elementær Geometri. 1ste Bog. Kjøbenhavn, Gjellerup, 1916. 8vo. 135 pp. Kr. 2.00

JADANZA (N.). Tavole tacheometriche centesimali. 4a edizione stereotipa. Torino, V. Bona, 1918. 8vo. 4 + 64 pp. L. 3.50

MÉREL (E.). Pour s'amuser dans la tranchée, to pass time in the trenches . . . 16 démonstrations du théorème de Pythagore. Paris et Nancy, Berger-Levrault, 1916. 8vo. 22 pp.

MORGAN (F. M.). Essentials of algebra and geometry. Six lessons preparatory to the study of trigonometry. New York, National War Work Council of Young Men's Christian Association, 1919. 12mo. 58 pp.

SCHUBERT (H.). Arithmetik und Algebra. (Sammlung Göschen, Nr. 47.) 2te durchgesehene Auflage. 7ter Neudruck. Berlin, Göschen, 1917. 8vo. 171 pp. M. 1.00

SCHÜLKE (A.). Vierstellige Logarithmentafeln für den Schulgebrauch zusammengestellt. 10te Auflage. Leipzig, Teubner, 1917. 10 + 22 pp. M. 0.70

TAYLOR (E. H.) and ALLEN (F.). Junior high school mathematics. New York, Holt, 1919. 12mo. 9 + 208 pp. $0.80

III. APPLIED MATHEMATICS.

ABRAHAM (M.). Theorie der Elektrizität. 1ter Band: Einführung in die Maxwellsche Theorie der Elektrizität. 5te umgearbeitete Auflage. Leipzig, 1918. M. 13.00

ANNUAIRE ASTRONOMIQUE et météorologique pour 1919. Par C. Flammarion. 55e année. Paris, Flammarion, 1919. 364 pp. Fr. 3.50

ANNUAIRE pour l'an 1919 publié par le Bureau des Longitudes. Avec des notices scientifiques de P. Appell et M. Hamy. Paris, Gauthier-Villars, 1919. 8 + 524 + 60 + 27 + 69 pp. Fr. 3.35

APPELL (P.). See ANNUAIRE and BUGAT-PUJOL.

ASHLEY (R. H.). Chemical calculations. 2d edition revised. New York, Van Nostrand, 1918. 12mo. 9 + 276 pp.

BALL (J.). Notes on the working of the "new navigation." Cairo, Government Press, 1918.

BLOCH (W.). Einführung in die Relativitätstheorie. (Aus Natur und Geisteswelt, Nr. 618.) Leipzig, Teubner, 1918. 16mo. 100 pp. M. 1.50

BORDIGA (G.). Matematica finanziaria. Primo corso (genesi dei numeri, calcoli numerici, etc.). Padova, A. Milani, 1918. 8vo. 231 pp. L. 10.00

BOUASSE (H.). Géographie mathématique. Paris, Delagrave, 1919. 8vo. 494 pp. Fr. 25.00

BRILL (A.). Das Relativitätsprinzip. Eine Einführung in die Theorie. 3te Auflage. Leipzig, Teubner, 1918. 8vo. 48 pp. M. 2.00

BUGAT-PUJOL (—.). Statique graphique. Préface de M. Paul Appell. Paris, Dunod et Pinat, 1918. 4to. 12 + 237 pp. + atlas de 41 plates. Fr. 24.00

COMESSATTI (A.). See SEVERI (F.).

CONNAISSANCE des temps ou des mouvements célestes pour le méridien de Paris à l'usage des astronomes et des navigateurs pour 1920. Paris, Gauthier-Villars, 1918. 8vo. 30 + 756 pp. Fr. 5.00

EDDINGTON (A. S.). Report on the relativity theory of gravitation. London, Physical Society of London, 1918. 7 + 91 pp. 6s.

EMSLEY (H. H.). See GLEICHEN (A.).

FAWDRY (R. C.). Dynamics. Part 2. (Bell's Mathematical Series.) London, Bell, 1919. Pp. 8 + 179–355 + 7. 2s. 6d.

FLAMMARION (C.). See ANNUAIRE ASTRONOMIQUE.

FREUNDLICH (—.). Die Grundlagen der Einsteinischen Gravitationstheorie. 2te Auflage. Berlin, Springer, 1917. M. 3.60

GLEICHEN (A.). The theory of modern optical instruments. A reference book for physicists, manufacturers of optical instruments, and for officers in the army and navy. Translated from the German by A. Gleichen, H. H. Emsley and W. Swaine. With an appendix on range finders. London, H. M. S. O., 1918. 12 + 376 pp.

HAMY (M.). See ANNUAIRE.

HOYLE (B.). Standard tables and equations in radio-telegraphy. London, Wireless Press, 1919. 14 + 159 pp. 9s.

ISING G.). Undersökningar rörande elektrometrar. Lund, Gleerupska Univ.-Bokhandeln, 1917. 8vo. 173 pp. Kr. 4.00

KEMP (P.). Alternating current electrical engineering. London, Macmillan, 1918. 9 + 494 pp. 17s.

LEVI-CIVITA (T.). Corso di mecanica razionale. Padova, tip. Universitaria, 1918. 3 volumes. 339 + 719 pp. L. 39.00

MARCOLONGO (R.). Il problema dei tre corpi da Newton (1686) ai nostri giorni. (Manuali Hoepli.) Milano, Hoepli, 1918. 24mo. 7 + 166 pp. L. 4.50

NEUENDORFF (R.). Praktische Mathematik. 1ter Teil: Graphische Darstellungen. Verkürztes Rechnen. Das Rechnen mit Tabellen. Mechanische Rechenhilfsmittel. Kaufmännisches Rechnen im täglichen Leben. Wahrscheinlichkeitsrechnung. 2te Auflage. Leipzig, Teubner, 1917. 4 + 106 pp. M. 1.20

PAGE (V. W.). The A B C of aviation. London, Crosby Lockwood, 1918. 274 pp. + 7 plates. 12s. 6d.

SEVERI (F.). Lezioni di geometria descrittiva, dettate nella r. università di Padova e nella libera università di Ferrara e raccolte dall' assistente prof. A. Comessatti. Padova, tip. Universitaria, 1918. 8vo. 12 + 728 pp. con tavole. L. 29.00

STOLTZ (G.). Utjämning av sjutton svenska livförsäkringsbolags dodlighetstabeller för läkareundersökta livförsäkrade. Stockholm, K. L. Beckman, 1917. 8vo. 5 + 210 pp.

SWAINE (W.). See GLEICHEN (A.).

VATER (R.). Die Dampfmaschine. I: Wirkungsweise des Dampfes im Kessel und in der Maschine. 4te Auflage. II: Ihre Gestaltung und Verwendung. 2te Auflage. Leipzig, Teubner, 1918. M. 1.50 + 1.50

———. Hebezeuge. Hilfsmittel zum Heben fester, flüssiger und gasförmiger Körper. 2te Auflage. Leipzig, Teubner, 1918. M. 1.50